

MATERI 13. VEKTOR

- Ruang Berdimensi n Euclidean
- Transformasi Linier R^n ke R^m

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Ruang-n Euclidean

(Euclidean n-space)

Review: *Bab 3 membahas Ruang-2 dan Ruang-3*

Ruang-n : himpunan yang beranggotakan vektor-vektor dengan n komponen

$$\{ \dots, \mathbf{v} = (v_1, v_2, v_3, v_4, \dots, v_n), \dots \}$$

- Atribut: arah dan “panjang” / norma $\|\mathbf{v}\|$
- Aritmatika vektor-vektor di Ruang-n:
 1. Penambahan vektor
 2. Perkalian vektor dengan skalar
 3. Perkalian vektor dengan vektor

Norma sebuah vektor:

Norma Euclidean (*Euclidean norm*) di Ruang-n :

$$\mathbf{u} = (u_1, u_2, u_3, \dots, u_n)$$

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2 + \dots + u_n^2}$$

Penambahan vektor: di Ruang-n

$$\mathbf{u} = (u_1, u_2, u_3, \dots, u_n); \quad \mathbf{v} = (v_1, v_2, v_3, \dots, v_n)$$

$$\mathbf{w} = (w_1, w_2, w_3, \dots, w_n) = \mathbf{u} + \mathbf{v}$$

$$\mathbf{w} = (u_1, u_2, u_3, \dots, u_n) + (v_1, v_2, v_3, \dots, v_n)$$

$$\mathbf{w} = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \dots, u_n + v_n)$$

$$w_1 = u_1 + v_1$$

$$w_2 = u_2 + v_2$$

.....

$$w_n = u_n + v_n$$

Negasi suatu vektor:

$$\mathbf{u} = (u_1, u_2, u_3, \dots, u_n)$$

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \dots, -u_n)$$

Selisih dua vektor:

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

$$= (u_1 - v_1, u_2 - v_2, u_3 - v_3, \dots, u_n - v_n)$$

Vektor nol: $\mathbf{0} = (0_1, 0_2, 0_3, \dots, 0_n)$

Perkalian skalar dengan vektor:

$$\mathbf{w} = k\mathbf{v} = (k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3, \dots, k\mathbf{v}_n)$$

$$(w_1, w_2, w_3, \dots, w_n) = (k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3, \dots, k\mathbf{v}_n)$$

$$w_1 = k\mathbf{v}_1$$

$$w_2 = k\mathbf{v}_2$$

.....

$$w_n = k\mathbf{v}_n$$

Perkalian titik: (perkalian Euclidean)

$$\mathbf{u} \cdot \mathbf{v} = \text{skalar}$$

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 + \dots + u_nv_n$$

$$\mathbf{u} \cdot \mathbf{v} = 0 \text{ jika } \mathbf{u} \text{ dan } \mathbf{v} \text{ ortogonal}$$

Catatan: perkalian silang hanya di Ruang-3

Aritmatika vektor di Ruang-n:

Teorema 4.1.1.: u, v, w vektor-vektor di Ruang-n

k, l adalah skalar (bilangan *real*)

- $u + v = v + u$
- $(u + v) + w = u + (v + w)$
- $u + 0 = 0 + u = u$
- $u + (-u) = (-u) + u = 0$
- $k(lu) = (kl)u$
- $k(u + v) = ku + kv$
- $(k + l)u = ku + lu$
- $1u = u$

Teorema 4.1.2:

Vektor-vektor $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ di Ruang- n ; k adalah skalar

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
- $\mathbf{v} \cdot \mathbf{v} > 0$ jika $\mathbf{v} \neq \mathbf{0}$
 $\mathbf{v} \cdot \mathbf{v} = 0$ jika dan hanya jika $\mathbf{v} = \mathbf{0}$

Teorema 4.1.3 - 4.1.5:

$$| \mathbf{u} \cdot \mathbf{v} | \leq \| \mathbf{u} \| \| \mathbf{v} \|$$

$$\| \mathbf{u} \| \geq 0$$

$$\| \mathbf{u} \| \geq 0 \text{ jika dan hanya jika } \mathbf{u} = \mathbf{0}$$

$$\| k\mathbf{u} \| = |k| \| \mathbf{u} \|$$

$$\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$$

$$d(\mathbf{u}, \mathbf{v}) \geq 0$$

$$d(\mathbf{u}, \mathbf{v}) = 0 \text{ jika dan hanya jika } \mathbf{u} = \mathbf{v}$$

$$d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$$

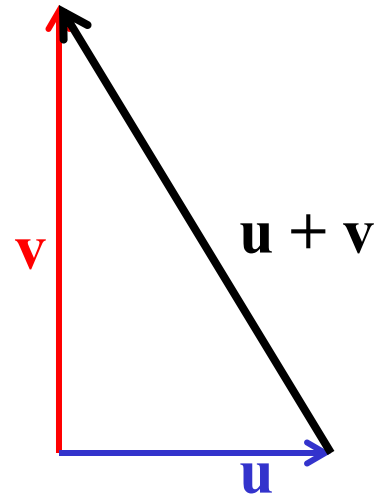
$$d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$$

Teorema 4.1.6 – 4.1.7:

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4} \|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4} \|\mathbf{u} - \mathbf{v}\|^2$$

Teorema Pythagoras

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$



Perkalian Titik (*dot product*) dikerjakan dengan perkalian matriks

$$\mathbf{u} = (u_1, u_2, u_3, \dots, u_n); \quad \mathbf{v} = (v_1, v_2, v_3, \dots, v_n)$$

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \dots + u_n v_n$$

Kalau vektor $\mathbf{u}$ dan vektor $\mathbf{v}$ masing-masing ditulis dalam notasi matriks baris, maka

$$\mathbf{u} \cdot \mathbf{v} = (u_1 \ u_2 \ u_3 \ \dots \ u_n) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{u}) (\mathbf{v})^T$$

$$\mathbf{u} = (u_1, u_2, u_3, \dots, u_n); \quad \mathbf{v} = (v_1, v_2, v_3, \dots, v_n)$$

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \dots + u_n v_n$$

Kalau vektor $\mathbf{u}$ dan vektor $\mathbf{v}$ masing-masing ditulis dalam notasi matriks kolom, maka

$$\mathbf{u} \cdot \mathbf{v} = (u_1 \ u_2 \ u_3 \ \dots \ u_n) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = \mathbf{v} \cdot \mathbf{u}$$

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{u})^T (\mathbf{v})$$

$$\mathbf{v} \cdot \mathbf{u} = (\mathbf{v})^T (\mathbf{u})$$

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{v})^T (\mathbf{u})$$

Matriks $\mathbf{A}$ ($n \times n$), $\mathbf{u}$ dan $\mathbf{v}$ masing-masing vektor kolom

$$\mathbf{A}\mathbf{u} \cdot \mathbf{v} = (\mathbf{A}\mathbf{u}) \cdot \mathbf{v} \stackrel{?}{=} \mathbf{u} \cdot (\mathbf{A}^T\mathbf{v})$$

$$\mathbf{u} \cdot \mathbf{A}\mathbf{v} \stackrel{?}{=} \mathbf{A}^T\mathbf{u} \cdot \mathbf{v}$$

$$\begin{aligned}(\mathbf{A}\mathbf{u}) \cdot \mathbf{v} &= \mathbf{v} \cdot (\mathbf{A}\mathbf{u}) \\&= \mathbf{v}^T(\mathbf{A}\mathbf{u}) \\&= (\mathbf{v}^T\mathbf{A})\mathbf{u} \\&= (\mathbf{A}^T\mathbf{v})^T \mathbf{u} \\&= \mathbf{u} \cdot (\mathbf{A}^T\mathbf{v})\end{aligned}$$